Evaluating the Steady State Distribution of Cumulative Reward in Hidden Markov Models *

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Abstract. *Hidden Markov Models (HMMs) have been widely used in the literature for modeling computer systems, for describing and predicting the loss and delay packet processes over the Internet and for solving network planning/dimensioning problems. This paper proposes a system of differential equations for calculating the distribution of the cumulative reward, when HMMs are used. Furthermore, we propose an iterative algorithm for obtaining an approximated solution for the differential equations. The overall technique for calculating the measure of interest is numerically robust and has a significant smaller computational cost when compared with approaches found in literature.*

1. Introduction

Markov reward models are a widely used mathematical tool in computer systems performance evaluation. In these models, rate rewards are associated with the states of a Markov chain or impulse rewards may be associated with transitions between states. In traditional Markov rate reward models, there is a one-to-many mapping between a reward value (from the set of all possible values) and the states in the model. In this framework, there are efficient algorithms in the literature to calculate many different measures of interest. Examples of such measures are: distribution of the cumulative reward over a finite interval of time, or in steady state, mean time to accumulate a given reward value, etc. (See [Anick et al. 1982], [Mogens Bladt and Sericola 2002], [de Souza e Silva and Gail 1998], [Leão et al. 2001], [Nabli and Sericola 1996], [Mahevas and Rubino 2001], [da Silva and Rubino 2006], [de Souza e Silva et al. 1995] and the references therein).

The type of models mentioned above and the major random variable of interest, can be formally defined as follows. Consider a homogeneous continuous-time Markov process $\mathcal{X} = \{X(t); t \geq 0\}$ with finite state space $\mathcal{S} = \{s_i; i = 1, \cdots, M\}$. To each state $s \in \mathcal{S}$, we assign a reward rate from a given set of reward values $\mathcal{R} = \{r_1, \cdots, r_K\}$. The random variable $IR(t)$, the instantaneous reward at time t, is $IR(t) = r_{c(s)}$ if $X(t) = s_i$, where $c(s)$ is index of the rate reward value associated with state s_i. The cumulative reward during an observation period $(0, t)$ is:

$$CR(t) = \int_0^t IR(\tau) d\tau.$$

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Calculating statistics for the random variable $CR(t)$, such as its steady state distribution, $P[CR \leq y] = \lim_{t\to\infty} P[CR(t) \leq y]$, is a valuable though not easy problem, since $CR(t)$ can be associated with important quantities of interest in many different areas. One of these is *fluid modeling*, which is a powerful tool for solving computer network models, both analytically or via simulation. In most packet models of networks with high speed links, the rate at which packets are generated is orders of magnitude higher than the rate of changes of these generation rates. Therefore, one can consider the packet traffic as a continuous flow with rates that change over time, governed by some stochastic process. The buffer that queues packets for transmission can be thought of a bucket that stores fluid at rates that vary according to, for instance, a Markovian model. Several papers propose algorithms that calculate $P[CR \leq y]$ efficiently when each state of a fluid model is associated with a single fluid rate [Anick et al. 1982], [Ahn and Ramaswami 2003], [da Silva Soares and Latouche 2003], [da Silva et al. 2004].

More recently, Hidden Markov Models (HMMs) have shown to be very useful in computer network modeling, besides the huge number of applications in other areas [Duarte et al. 2003], [Wei et al. 2002], [Filho et al. 2006], [Salamatian and Vaton 2001], [de Souza e Silva et al. 2004]. For instance, HMMs [Rabiner 1989] have been applied to traffic modeling and to model the packet loss rate process in a network path, to cite just a few applications. Briefly, a Hidden Markov Model is composed of two coupled stochastic processes. The first is a Markov chain and the second is a process that generates observable symbols as time evolves. The probability distribution of generating one particular symbol of a given set at time t is fully determined by the current state of the chain. Note that one can think of the observable discrete symbols as rate values associated with the model states.

This paper is concerned with the evaluation of the steady state distribution of cumulative reward in probabilistic functions of Markov chains, for instance, when HMMs are used to model traffic. The contribution of our work can be summarized as follows: (i) we show that the measure $P[CR \leq y] = \lim_{t\to\infty} P[CR(t) \leq y]$ for an HMM can be calculated using a set of differential equations that it is a generalization of the well-known traditional set of equations (e.g. [de Souza e Silva and Gail 2000]), for the case of traditional fluid (or reward) models; (ii) we develop an iterative algorithm for obtaining an approximate solution of the system of differential equations proposed.

The remainder of the paper is organized as follows. In Section 2 we present the notation we use and briefly survey known results that are useful in the development of our algorithm. Section 3 obtains the system of differential equations for calculating the distribution of *CR*. In Section 4, we present our iterative approximation algorithm for solving the differential equations. An application example, useful for network dimensioning is presented in Section 5. Section 6 summarizes our contribution.

2. Notation and Background Material

In this section we present the notation used throughout the paper and the background needed. In what follows, matrices are denoted by uppercase boldface letters and each corresponding lowercase letter represent an entry in the matrix. Vectors will be denoted by lowercase boldface letters. $\mathbf{A}_{i*}$ and $\mathbf{A}_{*j}$ denotes the i^{th} row and j^{th} column of matrix $\mathbf{A}$, respectively. We use the symbol "$\bullet$" to indicate an element-wise vector multiplication.

We begin by defining a Hidden Markov Model, based on [Rabiner 1989]. Briefly, a Hidden Markov Model is composed of two coupled stochastic processes. The first is a Markov chain and the second is an observation process whose distribution at any given time is fully determined by the current state of the chain.

Let $\mathcal{X}^* = \{X^*(k); k \geq 0\}$ be a discrete-time Markov chain, with state space set defined by $\mathcal{S}^* = \{s_1, s_2, \cdots, s_{|\mathcal{S}^*|}\}$ and cardinality $|\mathcal{S}^*|$. In our notation, the process $\mathcal{X}^*$ is the discrete equivalent of a continuous-time Markov process $\mathcal{X} = \{X(t); t \geq 0\}$, after the uniformization technique is applied. (Details can be found in [de Souza e Silva and Gail 2000].). The initial state distribution is given by the $|\mathcal{S}^*|$-dimensional vector $\boldsymbol{\pi}(0)$, with $\pi_i(0) = P[X^*(0) = i]$. The state transition probabilities are given by the $|\mathcal{S}^*| \times |\mathcal{S}^*|$ matrix $\mathbf{P} = \{p_{ij}\}$, where $p_{ij} = P[X^*(k) = j \,|\, X^*(k-1) = i]$.

The observation process is denoted by $\mathcal{Y} = \{Y(k); k \geq 0\}$, with a total of $|\mathcal{R}|$ emission symbols that belong to the set $\mathcal{R} = \{r_1, r_2, \cdots, r_{|\mathcal{R}|}\}$, controlled by the $|\mathcal{S}| \times |\mathcal{R}|$ matrix $\mathbf{\Gamma} = \{\gamma_{ij}\}$, where $\gamma_{ij} = P[Y(k) = j \,|\, X^*(k) = i]$. In this paper, each symbol corresponds to a *rate reward* value. The stochastic process $\mathcal{Y}$ is called an observable Markov model and is completely determined by the process $\mathcal{X}^*$. For convenience, we use the compact notation $\theta = (\mathbf{P}, \mathbf{\Gamma}, \boldsymbol{\pi})$ to indicate the complete parameter set of a Hidden Markov model. A comprehensive survey on HMMs is [Rabiner 1989].

Let $\mathcal{W} = \{W(k); k \geq 0\}$ be a discrete-time Markov process, with the set of states $\mathcal{T} = \{t_1, t_2, \cdots t_{|\mathcal{S}^*| \times |\mathcal{R}|}\}$ and cardinality $|\mathcal{S}^*| \times |\mathcal{R}|$. The i^{th} state of $\mathcal{W}$ belongs to the Cartesian product of sets $\mathcal{S}$ and $\mathcal{R}$. The probability matrix that governs the state transitions is denoted by $\hat{\mathbf{P}} = \{\hat{p}_{i,(j,s)}\}$, where $\hat{p}_{i,(j,s)} = p_{ij}\gamma_{js}$. We refer to process $\mathcal{W}$ as the expanded version of processes $\mathcal{X}^*$ and $\mathcal{Y}$.

Using a straightforward argument (based in part on Section III of [Rabiner 1989]) Lemma 1 shows that process $\mathcal{X}^*$ is equivalent to the expanded process $\mathcal{W}$, given the sequence of rewards emitted by the observation process $\mathcal{Y}$. This equivalence is essential to analyze the set of differential equations presented in Section 3.

Lemma 1 *Let Ω be an observation sequence, with length K, $\Omega = \{r_1, r_2, \cdots, r_K\}$, of the Hidden Markov Model θ and $P[\Omega \,|\, \theta]$ and $P[\Omega \,|\, \mathcal{W}]$ be the probability of occurrence of the sequence Ω, given the model θ and the expanded process $\mathcal{W}$, respectively. Then $P[\Omega \,|\, \theta] = P[\Omega \,|\, \mathcal{W}]$.*

Proof: Consider the state sequence: $\pi = \{s_1, s_2, \cdots, s_K\}$. Let $c(k)$ be the function that gives the index of one of possible reward rate values associated with state s_k. (In the expanded process, only one reward rate is associated to a state.) From our notation, recall that, if the process is in state s_k, the reward $r_{c(k)}$ is observed with probability $\gamma_{s_k r_{c(k)}}$. Then,

$$P[\Omega \,|\, \pi, \theta] = \prod_{k=1}^{K} P[r_{c(k)} \,|\, s_k, \theta] = \gamma_{s_1 c(1)} \gamma_{s_2 c(2)} \cdots \gamma_{s_K c(K)}, \tag{1}$$

where the last equality is true since an observation depends only on the current state and the emission probability at that state.

The probability of a such state sequence π is:

$$P[\pi \,|\, \theta] = \pi_{s_1} p_{s1s2} p_{s2s3} \cdots p_{s_{K-1}s_K}. \tag{2}$$

Therefore:

$$P[\Omega \mid \theta] = \sum_{\forall \pi \in \mathcal{P}} P[\Omega \mid \pi, \theta] P[\pi \mid \theta] = \sum_{s_1 s_2 \cdots s_K} \pi_{s_1} \gamma_{s_1 r_{c(1)}} p_{s_1 s_2} \gamma_{s_2 r_{c(2)}} \cdots p_{s_{K-1} s_K} \gamma_{s_K r_{c(K)}}. \tag{3}$$

The probability $P[\Omega \mid \mathcal{W}]$ can be obtained in the same way, but nothing that, in state s_k, $\gamma_{s_K r_{c(K)}} = 1$. Recall that $\hat{p}_{s_k,(s_k,r_{c(k)})}$ is obtained from the original HMM and from the emission probabilities at s_k. Then,

$$P[\Omega, m \mid \mathcal{W}] = \pi_{s_1} \hat{p}_{s_1,(s_2,r_{c(2)})} \hat{p}_{s_2,(s_3 r_{c(3)})} \cdots \hat{p}_{s_{K-1},(s_K r_{c(K)})}. \tag{4}$$

By the definition of $\hat{p}_{i,(j,s)}$, equation (4) is rewritten as:

$$P[\Omega, m \mid \mathcal{W}] = \pi_{s1} \gamma_{s_1 r_{c(1)}} p_{s_1 s_2} \gamma_{s_2 r_{c(2)}} p_{s_2 s_3} \gamma_{s_3 r_{c(3)}} \cdots p_{s_{K-1}} \gamma_{s_K r_{c(K)}}.$$

Summing over all paths in $\mathcal{M}$, the Lemma is established. □

In what follows we introduce a few results on systems of differential equations that we will use later in this paper. Let:

$$\frac{d\mathbf{x}(y)}{dy} = \mathbf{A}\mathbf{x}(y) + \mathbf{g}(y) \tag{5}$$

be a general nonhomogeneous system, as presented in [Boyce and Diprima 2004] (chapter 7).

Define $\mathbf{T}$ as the matrix whose columns are equal to the eigenvectors $\{\boldsymbol{\xi}^{(1)}, \cdots, \boldsymbol{\xi}^{(n)}\}$ of another matrix $\mathbf{A}$, with dimension $n \times n$, and $\boldsymbol{\zeta}(y)$ is a vector with a set of dependent variables, written as:

$$\mathbf{x}(y) = \mathbf{T}\boldsymbol{\zeta}(y). \tag{6}$$

Substituting Equation (6) into (5) and multiplying the result equation by $\mathbf{T}^{-1}$, the following system of differential equations is obtained:

$$\frac{d\boldsymbol{\zeta}(y)}{dy} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\boldsymbol{\zeta}(y) + \mathbf{T}^{-1}\mathbf{g}(y) = \mathbf{D}\boldsymbol{\zeta}(y) + \mathbf{h}(y), \tag{7}$$

where $\mathbf{h}(y) = \mathbf{T}^{-1}\mathbf{g}(y)$ and $\mathbf{D}$ is a diagonal matrix whose the diagonal elements are the eigenvalues $\{r_1, \cdots, r_n\}$ of $\mathbf{A}$, for the cases where matrix $\mathbf{A}$ is diagonalizable. Equation (7) is a system with n independent equations for the $\{\zeta_1(y), \cdots, \zeta_n(y)\}$ variables that can be solved independently for each $j = \{1, \cdots, n\}$:

$$\frac{d\zeta_j(y)}{dy} = r_j \zeta_j(y) + h_j(y), \tag{8}$$

where $h_j(y)$ is a linear combination of $\{g_1(y), \cdots, g_n(y)\}$.

Equation (8) is a first order linear equation, with the following solution:

$$\zeta_j(y) = e^{\lambda_j y} \int_{y_0}^{y} e^{-\lambda_j s} h_j(s) ds + c_j e^{\lambda_j y}, \quad j = 1, \cdots, n \tag{9}$$

where c_j is an arbitrary constant known as constant of integration. The solution of equation (5) is obtained by (6) and (9).

3. The Distribution of the Cumulative Reward

In this section, we derive a set of differential equations for $P[CR \leq y]$, when a HMM is used, as defined in previous section. Let $\mathcal{X} = \{X(t); t \geq 0\}$ be a continuous-time Markov chain with a finite state space $\mathcal{S} = \{s_1, s_2, \cdots, s_{|\mathcal{S}|}\}$ with cardinality $|\mathcal{S}|$, and generator matrix $\mathbf{Q} = \{q_{ij}\}$. As before, $\mathcal{Y} = \{Y(t); t \geq 0\}$ is defined as a probabilistic function of $\mathcal{X}$ with state space in the set $\mathcal{R} = \{r_1, r_2, \cdots, r_{|\mathcal{R}|}\}$, with $|\mathcal{R}|$ elements. We transform $\mathcal{X}$ into a discrete-time Markov chain through uniformization. Let Λ be the uniformization rate, where $\Lambda \geq \max_i |q_{ii}|$, and therefore, $\mathbf{P} = \mathbf{I} + \mathbf{Q}/\Lambda$ is the discrete-time transition probability matrix.

In the uniformized system, $Y(t) = l$ with probability γ_{jl} given that $X(t) = j$. Recall that the value of $Y(t)$ determines the current rate $r_{Y(t)}$. The cumulative reward at t is, as introduced in Section 1,

$$CR(t) = \int_0^t r_{Y(t)} dt. \tag{10}$$

Let $\mathbf{F}(y,t)$ be the joint distribution matrix with the (i,j)-th element $F_{ij}(y,t) = P[\boldsymbol{CR}(t) \leq y, X(t) = j \,|\, X(0) = i]$. Let $\mathbf{W}(y,t)$ be the corresponding density matrix, $W_{ij}(y,t) = P[\boldsymbol{CR}(t) = y, X(t) = j \,|\, X(0) = i]$ [1]. Let $\mathbf{F}^l(y,t)$ be the matrix with entries $F_{ijl}(y,t) = P[\boldsymbol{CR}(t) \leq y, X(t) = j, Y(t) = l | X(0) = i]$ and $\mathbf{W}^l(y,t)$ the corresponding density matrix.

In what follows we obtain a partial differential equation for $\mathbf{W}(y,t)$ following similar steps as in [de Souza e Silva and Gail 2000]. (Due to the lack of space, we refer the reader to [da Silva 2006] for more details about the derivation of this equation.) Using Bayes' rule we obtain:

$$\begin{aligned} W_{ijl}(y,t) = \sum_{k=1}^{|\mathcal{S}|} \sum_{n=1}^{|\mathcal{R}|} \int_z & P[\boldsymbol{CR}(t) = y, X(t) = j, Y(t) = l | \boldsymbol{CR}(t-h) = z, X(t-h) = k, Y(t-h) = n, X(0) = i] \\ & P[\boldsymbol{CR}(t-h) = z, X(t-h) = k, Y(t-h) = n | X(0) = i] dz. \end{aligned} \tag{11}$$

If h is sufficiently small, there is at most one state transition in $(t-h, t]$. From the Markov property and nothing that $\boldsymbol{CR}(t)$ is independent of $X(t)$ and $Y(t)$, given $X(t-h)$ and $Y(t-h)$:

$$\begin{aligned} W_{ijl}(y,t) = \sum_{k=1}^{|\mathcal{S}|} \sum_{n=1}^{|\mathcal{R}|} \int_z & P[\boldsymbol{CR}(t) = y | \boldsymbol{CR}(t-h) = z, X(t-h) = k, Y(t-h) = n] \\ & P[X(t) = j, Y(t) = l | X(t-h) = k, Y(t-h) = n] W_{ikn}(z, t-h) dz. \end{aligned} \tag{12}$$

For h small, $\boldsymbol{CR}(t) = \boldsymbol{CR}(t-h) + r_{Y(t-h)} h + o(h)$ (where $o(h)$ is an error function which converges to 0 faster than h) and then:

$$P[\boldsymbol{CR}(t) = y | \boldsymbol{CR}(t-h) = z, X(t-h) = k, Y(t-h) = n] = \delta(y - (z + h r_n) + o(h)), \tag{13}$$

[1] although $\boldsymbol{CR}(t)$ is a continuous random variable, we use the notation $\boldsymbol{CR}(t) = y$ to facilitate the exposition

where $\delta()$ is the Dirac delta function.

In Equation (12) note that $P[X(t) = j, Y(t) = l | X(t-h) = k, Y(t-h) = n]$ is the transition probability of the expanded version of original HMM as defined in the previous section. Since state transitions of the uniformized process occur at a Poisson rate Λ, after some algebraic manipulations and taking the limit as $h \to 0$ and $t \to \infty$, we obtain:

$$\frac{d\mathbf{w}^l(y)}{dy} = \mathbf{w}(y)\left[\mathbf{Q} + \Lambda\mathbf{I}\right] \operatorname{diag}\left\{\frac{\gamma_{jl}}{r_l}\right\}_j - \frac{\Lambda}{r_l}\mathbf{w}^l(y). \tag{14}$$

where vectors $\mathbf{w}^l(y)$ and $\mathbf{w}(y)$ are each one of the identical lines of matrices $\mathbf{W}^l(y,t)$ and $\mathbf{W}^l(y,t)$ as $t \to \infty$.

Define $\mathbf{R}^l$ as the diagonal matrix $\operatorname{diag}\{\gamma_{jl}/r_l\}_j$. Then $\mathbf{f}^l(y)$ and $\mathbf{f}(y)$, defined in the same way as $\mathbf{w}^l(y)$ and $\mathbf{w}(y)$ immediately follows from (14).

$$\frac{d\mathbf{f}^l(y)}{dy} = \mathbf{f}(y)\left[\mathbf{Q} + \Lambda\mathbf{I}\right]\mathbf{R}^l - \frac{\Lambda}{r_l}\mathbf{f}^l(y). \tag{15}$$

Summing over all rewards in the set $\mathcal{R}$ and defining $\mathbf{R} = \operatorname{diag}\{\sum_{\forall l}(\gamma_{jl}/r_l)\}_j$, we finally obtain:

$$\frac{d\mathbf{f}(y)}{dy} = \mathbf{f}(y)\left[\mathbf{Q} + \Lambda\mathbf{I}\right]\mathbf{R} - \Lambda\sum_{\forall l}\frac{\mathbf{f}^l(y)}{r_l}. \tag{16}$$

Solving equation (16) is equivalent to solving the set of $|\mathcal{R}|$ equations in (14). Although the set of equations of type (14) has a closed form solution written in terms of its spectral expansion (similarly to [Anick et al. 1982], for the case in which there is only one rate reward assigned to each state of the process under study), the computational complexity of calculating $\mathbf{f}^l(y)$ can be high if the process under study has several rate rewards associated with each hidden state. Furthermore, numerical errors are likely to occur in this case.

In order to reduce the computational cost and to solve the system of differential equations that describes the HMM under study, we propose an iterative algorithm that produces an approximation solution of (16). This algorithm will be presented in Section 4.

We still have to address the boundary conditions. A system of differential equations is not completely specified only by its equations, since the solution is a family of curves. Therefore, it is essential to apply the boundary conditions which are algebraic conditions on the values functions to obtain the final answer. However, this may not be a trivial problem [Press et al. 1997].

In this paper we are concerned with models where both lower and upper bounds on the cumulative reward random variable are specified. Let the lower bound be equal to 0 and the upper bound be equal to U. (For instance if *CR* models the content of a buffer, then clearly *CR* ≥ 0 and U is the buffer upper limit.)

We assume that all rate rewards associated with a state are either strictly positive, strictly negative or equal to zero. This is not a restrictive assumption and can always be satisfied by an appropriate mapping of states to rewards. In what follows, due to space

limitations, we do not consider the zero reward case, but it can be easily handled. We then divide the set of states into two types: $\mathcal{S}^+$, where all rate rewards emitted at that state are strictly positive and the complementary set $\mathcal{S}^-$, where all rate rewards associated with the states are strictly negative. The states that belong to the first set are called *filling states* and the states in the second one are *emptying states*.

The following boundary conditions are specified to the system of differential equations in (16) (for a detailed discussion on boundary conditions, we refer the reader to [Anick et al. 1982] and [Schwartz 1996]):

1. For $y = 0$ and $s_i \in \mathcal{S}^+$, we have $f_i(0) = 0$. This is true since, once in state s_i, the system can only accumulate fluid.
2. For $y = U^-$ and $s_i \in \mathcal{S}^-$, we have $f_i(U^-) = \pi_i$. This is true since in s_i the system is loosing fluid, so the accumulate reward (or buffer content) can not be at its highest value.

It is straightforward to see that the following system of equations is equivalent to (16) :

$$\frac{d[\mathbf{f}(y)]^T}{dy} = \mathbf{R}[\mathbf{Q} + \Lambda\mathbf{I}]^T[\mathbf{f}(y)]^T \; - \Lambda \sum_{\forall l} \left(\frac{[\mathbf{f}^l(y)]^T}{r^l} \right) \tag{17}$$

To simplify the notation, we refer to vectors $[\mathbf{f}^l(y)]^T$ and $[\mathbf{f}(y)]^T$ as $\mathbf{f}^l(y)$ and $\mathbf{f}(y)$, respectively.

4. The Iterative Algorithm

In this section we devise an iterative algorithm to obtain an approximate solution for the system of differential equations in (17). The basic idea is to first obtain an initial approximation for the solution. Then we try to refine the initial result by splitting the main differential equation into two. Each of those has a term that is assumed to be constant and obtained by solving the other equation in the pair. Both can be combined into a single equation with unknowns that are calculated from the solution in the preceding step. The iterative solution, for steps $k = 0, 1 \ldots$ is described next.

4.1. Initial Solution

In order to obtain an initial solution, we assume that the term:

$$\Lambda\mathbf{R}\mathbf{f}(y) - \Lambda \sum_{\forall l} \left(\frac{\mathbf{f}^l(y)}{r^l} \right)$$

in (17) is small enough to be neglected.

The element $f_j^l(y) = P[\mathit{CR} \leq y, X = j, Y = l]$ is equal to $P[\mathit{CR} \leq y, X = j]P[Y = l|\mathit{CR} \leq y, X = j] = f_j(y)P[Y = l|\mathit{CR} \leq y, X = j]$. The assumption used for calculating the initial solution is true if $P[Y = l|\mathit{CR} \leq y, X = j] \approx P[Y = l|X = j]$, that is if $\mathbf{f}^l(y) \approx \mathbf{f}(y) \bullet \boldsymbol{\gamma}^l$ (each element in $\boldsymbol{\gamma}^l$ is the probability of emitting the reward l in each hidden state). In other words, we are assuming that the rate reward emitted in state j is independent of the present value of the accumulated reward. A good approximation is expected if the rate reward values emitted in state j do not differ significantly.

Applying the above assumption to (17), the initial solution $\mathbf{f}^{(0)}(y)$ is estimated by solving the following system:

$$\frac{d\mathbf{f}^{(0)}(y)}{dy} = \mathbf{R}\mathbf{Q}^T\mathbf{f}^{(0)}(y). \tag{18}$$

It is well known that the solution of (18), $\mathbf{f}(y) = e^{\mathbf{R}\mathbf{Q}^T}$, can be written in terms of the spectral expansion of matrix $\mathbf{R}\mathbf{Q}^T$:

$$\mathbf{f}^{(0)}(y) = \sum_{\forall i \in \mathcal{S}} a_i \boldsymbol{\sigma}_i e^{\lambda_i y}.$$

It is important to emphasize that, in order to estimate $\mathbf{f}^{(0)}(y)$, it is necessary to obtain the eigenvalues and eigenvectors of a matrix with dimension $|\mathcal{S}| \times |\mathcal{S}|$. This contrasts with the computational complexity if we use the expanded process defined in Section 2. If the expanded process is used, the final solution requires the calculation of the eigenvalues and eigenvectors of a matrix with dimension $|\mathcal{S}| \times |\mathcal{W}|$, potentially a much larger matrix than the first. Furthermore, for the initial estimation, we have to solve a smaller linear system of equations for obtaining the set of constants of integration, given by the specified boundary conditions, as compared with the solution of the expanded process. In summary, using the approach presented in this section, the computational complexity and the probability of numerical errors are reduced for the cases where the cardinality of the Cartesian product of the sets $\mathcal{S}$ and $\mathcal{R}$ is high.

4.2. Estimating Vector $\mathbf{f}^{(l,k)}(y)$

The next step of the iterative algorithm is to estimate the value of $\mathbf{f}^{l}(y)$. Let us rewrite Equation (15) as:

$$\frac{d\mathbf{f}^{l}(y)}{dy} = \mathbf{R}^l\mathbf{Q}^T\mathbf{f}(y) + \frac{\Lambda}{r_l}[(\mathbf{f}(y) \bullet \boldsymbol{\gamma}^l)] - \frac{\Lambda}{r_l}\mathbf{f}^l(y). \tag{19}$$

For each distinct reward in set $\mathcal{R}$, we have to solve the system described in (19). As stated in Section 3, solving $|\mathcal{R}|$ linear systems of type (19) is equivalent of finding the desired solution using the expanded process $\mathcal{W}$, with state space cardinality equal $|\mathcal{S}| \times |\mathcal{W}|$.

Lemma 2 *For $l = \{1, 2, \cdots, |\mathcal{R}|\}$, we can estimate $\mathbf{f}^{(l,k)}(y)$ from:*

$$\mathbf{f}^{(l,k)}(y) = \phi^{(l,k)}(y) + \left(\frac{r_l}{\Lambda}\right)[\mathbf{R}^l\mathbf{Q}^T\mathbf{f}^{(k-1)}(y)] + (\mathbf{f}^{(k-1)}(y) \bullet \boldsymbol{\gamma}^l) \tag{20}$$

with $\phi_j^{(l,k)}(y) = \alpha_j^{(l,k)}(y)e^{\left(\frac{-\Lambda}{r_l}\right)y}$, $j = \{1, \cdots, |\mathcal{S}|\}$ and $\alpha_j^{(l,k)}(y)$ is the j^{th} constant of integration calculated using the boundary conditions specified in Section 3.

Proof: The proof follows immediately from the known solution of the nonhomogeneous system of differential equations of the same type of Equation (20). □

4.3. Estimating vector $\mathbf{f}^{(k)}(y)$

Using the estimation of $\mathbf{f}^{(l,k)}(y)$ above, it remains to estimate $\mathbf{f}^{(k)}(y)$.

Lemma 3 *Consider the estimation of* $\mathbf{f}^{(l,k)}(y)$ *from the previous lemma. Then:*

$$f_j^{(k)}(y) = \mathbf{\Upsilon}_{j*}\boldsymbol{\psi}^{(k)}(y) \quad j = \{1, \cdots, |\mathcal{S}|\}, \tag{21}$$

where $\mathbf{\Upsilon}$ *is such that its columns are equal to the eigenvalues of* $\mathbf{RQ}^T$*, and the elements of vector* $\boldsymbol{\psi}^{(k)}(y)$ *are*

$$\psi_1^{(k)}(y) = a_1^{(k)}(y) + y\mathbf{\Upsilon}_{1*}^{-1}\boldsymbol{\kappa}^{(k)}(y);$$

$$\psi_j^{(k)}(y) = a_j^{(k)}(y)e^{\lambda_j y} + \frac{1}{\lambda_j}\mathbf{\Upsilon}_{j*}^{-1}\boldsymbol{\kappa}^{(k)}(y) \quad j = \{2, \cdots, |\mathcal{S}|\}.$$

$a_j^{(k)}(y)$ *is the constant of integration of the* j^{th} *state, and*

$$\boldsymbol{\kappa}^{(k)}(y) = \Lambda\mathbf{R}\sum_{\forall l}\mathbf{f}^{(l,k)}(y) - \Lambda\sum_{\forall l}\left(\frac{\mathbf{f}^{(l,k)}(y)}{r_l}\right),$$

and $\mathbf{f}^{(k)}(y) = \sum_{\forall l}\mathbf{f}^{(l,k)}(y)$.

Proof: Similarly to the proof of Lemma 2, the lemma follows by identifying Equation (21) with a nonhomogeneous system of differential equations with known solution. □

The stopping rule for our iterative algorithm is as follows. We compute the norm of the difference of the results in consecutive iteration steps, $\mathbf{f}^{(k)}(y)$ and $\mathbf{f}^{(k-1)}(y)$. When the difference is smaller than a specified tolerance, the iterative procedure ends. The tests we performed indicate that this simple procedure works very well.

Below we address the computational complexity of the approximate algorithm. We present the total number of multiplications, without justification. The details are beyond the scope of this paper but can be found in [da Silva 2006]. We assume that the computational cost of calculating eigenvectors and eigenvalues of a matrix is $10M^3$ multiplications and that the cost of solving a linear system is M^3 multiplications, where M is the dimension of the associated matrix [Moler and Loan 1978].

The computational complexity is dominated by the following operations:

1. The initial solution estimation. The total cost is $11|\mathcal{S}|^3$ multiplications;
2. Solving for $\mathbf{f}^l(y)$. If c is the total number of points needed for the distribution, the total cost of this step is $c|\mathcal{R}|(2|\mathcal{S}|^2 + |\mathcal{S}|)$;
3. Obtaining $\mathbf{f}(y)$. First, we have to solve c systems of linear equations, with $|\mathcal{S}|^3$ multiplications. Then, it is necessary to perform $c(|\mathcal{S}|^3 + 2|\mathcal{S}|^2 + |\mathcal{S}|)$ multiplications.

5. Examples

This section illustrates an application of the algorithm proposed in this paper. We have assessed the efficiency of our iterative approximation algorithm running a hundred HMMs. The HMMs used either represented hypothetical systems or are traffic models of the main Internet link of the Computer and System Engineering Department at COPPE/UFRJ. The

traffic modeled by the HMMs are assumed to feed a finite buffer. In this application, the random variable *CR* is the number of bits in the buffer in steady state. Due to the lack of space, we present our results using two sets of experiments. The first set uses an empirical system, with a simple Hidden Markov model with two hidden states. The main purpose of this test is to show the accuracy of the iterative approximation algorithm, for different values of system load (ρ). For the second set of tests, a queue is fed by a HMM parameterized from real measurements performed at our main Department link. The value of the parameters were radomly chosen in order to study the accuracy of the proposed iterative approximation algorithm. We also briefly discuss the main results after applying our method for solving 170 hypothetical HMMs. We refer the reader to [da Silva 2006] for more examples.

In order to evaluate the accuracy of the method, we calculate two measures: (i) the maximum absolute error of $P[CR \leq y]$ and; (ii) the maximum relative error of $P[CR \leq y]$. Let $P[CR \leq y]$ and $\hat{P}[CR \leq y]$ be the exact and the approximated solutions, respectively. The measure (i) is $|P[CR \leq y] - \hat{P}[CR \leq y]|$, and (ii) is $|P[CR \leq y] - \hat{P}[CR \leq y]|/P[CR \leq y]$.

In the first set of experiments, the HMM used has two hidden states: A, with input rate $\delta_1 = 0$, and B, with input rates $\delta_2 = 10000$ and $\delta_3 = 15000$. We compared the result of our iterative approximation algorithm against the method presented in [Anick et al. 1982] using the equivalent expanded model in which states are associated with a single rate reward only (see Section 2).

The following set of parameters were used. The channel capacity is $\mu = 4713$, buffer size is $B = 500$ and $\rho = 0.99$. The reward values assigned are: state A: $r_1 = -4713$; and state B $r_2 = 5287$, with probability $p_2 = 0.2$, and $r_3 = 10287$, with probability 0.8. (Note that the input rate and channel capacity where adjusted to match the total utilization assuming that no losses occur.) The result is presented in Figure 1. The maximum absolute error is equal to $4.62876e$–3 only.

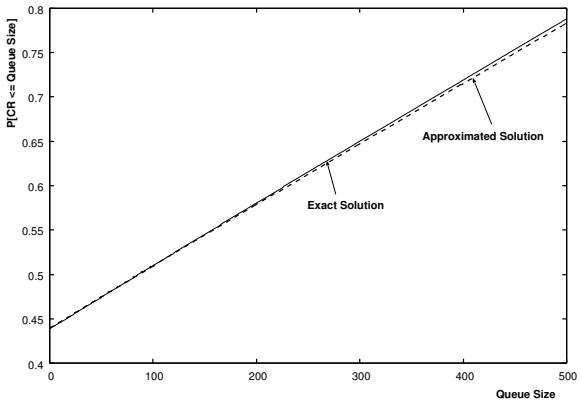

Figure 1. Approximated solution for $\rho = 0.99$ and $B = 500$.

We change the utilization to $\rho = 0.7$ (the channel capacity is then $\mu = 6665$), and used the reward values $\mathcal{R} = \{r_1 = -6665, r_2 = 3335, r_3 = 10334\}$ with the same probability distribution as in the previous example. The result is shown in Figure 2. In this case, the maximum absolute error is $1.5031e$–2.

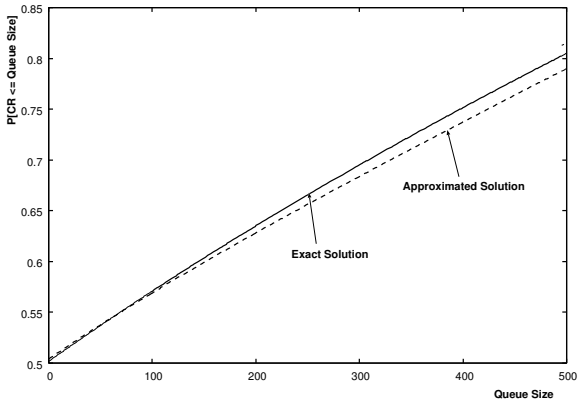

Figure 2. Approximated solution for $\rho = 0.7$ and $B = 500$.

We now increase the buffer size to $B = 1000$, and use $\mu = -8332$ with $\rho = 0.5$. The set of rate rewards is $\mathcal{R} = \{r_1 = -8332, r_2 = 1668, r_3 = 6668\}$. Figure 3 shows the result. The maximum absolute error is $2.8262e{-}2$. The examples indicate that the accuracy of our algorithm is good for different values of ρ.

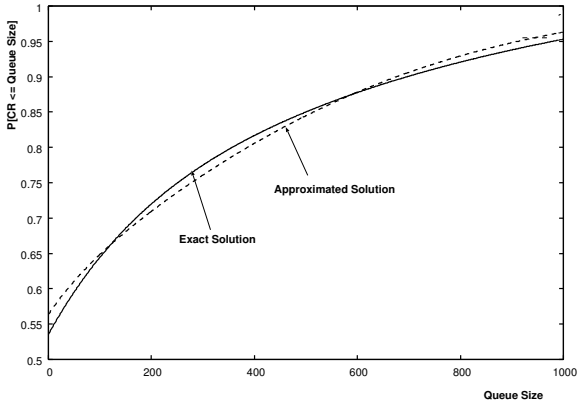

Figure 3. Approximated solution for $\rho = 0.5$ and $B = 1000$.

In the second set of examples, the rate rewards emitted by the observation process $\mathcal{Y}$ represent the traffic rates on our main Department link. The HMM was parameterized using real measures on that link. The HMM has four states. Let $|\mathcal{S}| = \{A, B, C, D\}$ be the set of hidden states and $|\mathcal{R}| = \{r_i \,|\, i = 1, \cdots, |\mathcal{R}|\}$ be the set of observation rate rewards. The buffer size is assumed to be equal to 1000. The following values are assigned to the rate rewards: $\mathcal{R} = \{r_1 = -821,\ r_2 = -663,\ r_3 = -505,\ r_4 = -347,\ r_5 = -189,\ r_6 = -189,\ r_7 = -347,\ r_8 = -821\, r_9 = 2181,\ r_{10} = 2655,\ r_{11} = 3919,\ r_{12} = 127,\ r_{13} = 285\, r_{14} = 601,\ r_{15} = 759\}$. The hidden state A emits $\{r_1, r_2, r_3, r_4, r_5\}$, with uniform distribution; state B emits the rate rewards $\{r_6, r_7, r_8\}$, with probability distribution $\{p_6 = 0.3,\ p_7 = 0.3,\ p_8 = 0.4\}$; state C emits $\{r_9, r_{10}, r_{11}\}$, with probabilities $\{p_9 = 0.3,\ p_{10} = 0.3,\ p_{11} = 0.4\}$ and state D emits the complementary set of rewards, with the following probability distribution: $\{p_{12} = 0.3,\ p_{13} = 0.2,\ p_{14} = 0.1,\ p_{15} = 0.4\}$.

In order to obtain the exact solution, we first applied the exact method of [Anick et al. 1982] to the extended model and calculated the eigenvalues and eigenvectors

of a matrix with dimension 15×15. However, numerical errors affected the calculation of the constants of integration and it was not possible to write the solution using its spectral expansion form. (As stated before numerical errors are common in moderate size models.) Then we used the method presented in [Leão et al. 2001] for calculating $P[CR(t) > y]$ for large values of t. Note that the cost of approach of [Leão et al. 2001] increases with t, and gives a result with bounded error provided that the convergence to steady state was reached for the value of t used. Figure 4 shows the approximated and exact solutions.

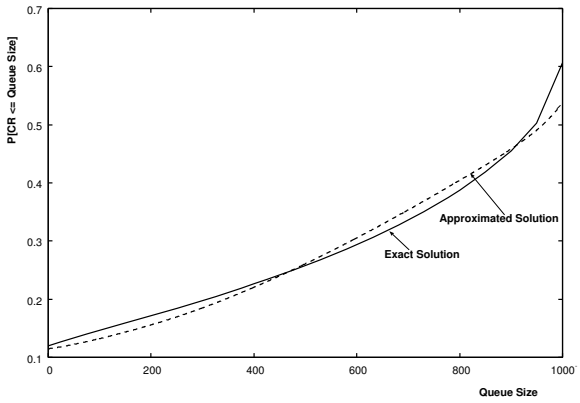

Figure 4. Approximated solution for the traffic model of our main Department link.

The proposed algorithm gives an approximated solution, with maximum absolute and approximate errors equal $6.7188e{-}2$ at $y = 1000$ and $1.1018e{-}1$, respectively. The main advantage of the proposed algorithm is the small computational complexity. Solving this problem by our method, we have to calculate the eigenvalues and eigenvectors of a matrix with dimension 4×4 only. The total number of iterations for reaching the stopping rule was 11 and a total of $8.2470e5$ multiplications were performed by the iterative algorithm. Using the method proposed in [Leão et al. 2001], it was necessary $2.3918e15$ multiplications in order to evaluate the exact solution for a large value of t.

We also have analyzed the results of 170 hypothetical HMMs. The rate reward value as well as the respective probabilities of emitting them were determined by using an uniformly distributed random generator. We observed that only 7% of the examples had numerical problems and from the subset without numerical problems, 90% had a maximum relative error less than 10%. Furthermore, the computational cost is smaller than the computational cost of the approaches in [Anick et al. 1982] and [Ahn and Ramaswami 2005] (The majority of the cases has a computational cost one order of magnitude smaller than the approaches cited above). Concerned with the convergence rate, the mean number of iterations for reaching the stopping rule was 6 iterations.

As a last remark, we observed that, the majority of examples with numerical problems have the following characteristics: the value of the rewards values differ from 4 orders of magnitude or the probability distributions associated with rewards in each state are very unbalanced.

6. Conclusion

We conclude by summarizing the main contributions of this paper. We have obtained a system of differential equations for calculating the steady state probability distribution of cumulative reward in Hidden Markov Reward Models. To our knowledge, this is the first algorithm that addresses the computation of this measure from a Hidden Markov Reward model. We showed that one can "expand" the state space of the HMM to obtain an equivalent Markov reward model. Traditional algorithms can be applied to solve the equivalent Markov reward model, but the computational cost is high.

For obtaining the system solution with a smaller computational cost than previous approaches that can solve the equivalent expanded Markovian reward model, we proposed an iterative approximation algorithm. As shown in Section 5, the accuracy of approximate solution is very good. Furthermore, our approach is susceptible to less numerical errors than traditional approaches applied to the extended model. This is due to the increase in the size of the (expanded) model as compared to the solution of the smaller Hidden Markov model.

We presented an example of a traffic model parameterized from measurements collected at our department link and some result of 170 hypothetical HMMs. We were able to accurately solve the overall queueing model.

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