Using Different Chord Lengths in Degree Three Chordal Rings and N2R Topologies

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Abstract. *Degree Three Chordal Rings and N2R Topologies are useful for physial and optical network topologies due to the combination of short distances, regularity and low degrees. In this paper we show how distances in terms of average distances and diameters can be significantly decreased by using chords of different lengths. These topologies are slightly less symmetric than the traditional ones, but the distances are virtually the same no matter which node in a given topology they are measured from.*

1. Introduction

The increasing use of computers and computer networks, in particular the Internet, has made societies all over the world highly depending on IT infrastructures. This dependency is supported by the fact that many different kinds of communication are converging towards a common IP-based platform[Madsen et al. 2002]; one general-purpose network can now be used for what previously required multiple dedicated networks such as those used for data, television and telephony. Furthermore, new applications are being designed to run in LAN as well as WAN environments, many which can by themself be categorized as critical with respect to QoS parameters and availability[Pedersen et al. 2005a][Lynch 2000]. Examples of emerging critical applications can be found within areas such as telemedicine and surveillance. The demands for availability and QoS are merely supported by todays best-effort IT/Broadband infrastructures, where software or hardware failures (including cable cuts) for most private customers will cause losses of connectivity.

In order to be able to design reliable Next Generation Networks, it is crucial to ensure sufficient levels of redundancy on the physical and optical levels. These levels are particularly important because they limit what guarantees can be provided by the higher layers: without the physical connectivity nothing can be done. It is especially interesting to look at the optical networks which are currently being deployed many places worldwide; these are often deployed over wide areas (such as backbone, distribution and access networks), and due to the high civil engineering costs they are extremely expensive to change once in the ground. Therefore the topologies must be carefully chosen.

Currently ring topologies are widely used because they offer full redundancy while still being simple to manage and implement both logically and physically. However, they are able to handle only a single failure, and large rings suffer from large distances resulting in large resource consumption especially when a failure occurs. Therefore, more robust topologies have recently got some attention. Symmetric or nearly symmetric networks[Dekker and Colbert 2004] of degree 3 seem to be particularly interesting because they offer up to three disjoint paths between any pair of nodes while still having a low degree, facilitating implementation and keeping costs down[Pedersen 2005]. Furthermore, the symmetry makes routing and restoration simple. Examples of such topologies are Degree Three Chordal Rings[Bujnowski et al. 2004], Honeycomb networks[Stojmenovic 1997] and $N2R$ structures[Madsen et al. 2003][Pedersen 2005]. A more comprehensive study of a larger number of topologies can be found in [Kotsis 1992], which also defines a large number of useful metrics. The $N2R$ structures have been introduced as a generalization of, and alternative to, Double Rings, to which they are superior in terms of average distance, diameter and several other parameters[Pedersen et al. 2004][Pedersen et al. 2005b][Frucht et al. 1971]. We recently described a generalization of the $N2R$ structures[Pedersen et al. 2006], where it was shown how to significantly reduce the distances by introducing two chord lengths instead of just one.

In this paper we extend the results of [Pedersen et al. 2006] by exploring the potentials of using three different chord lengths. While the way of constructing the modified $N2R$ is a rather straight-forward generalization of [Pedersen et al. 2006], the construction of modified Degree Three Chordal Rings is truly novel. To the best of our knowledge these topologies have not been studied before. The modified $N2R$ and Chordal Rings proposed can hold a virtually unlimited number of different chords, but in this paper we have calculated the results with only one, two and three chord lengths. The results are promising for both $N2R$ and Chordal Rings, but the most significant improvements can be found for the Chordal Rings.

The paper is organised as follows. Section 2 provides the basic terminology, notation and methods, and introduces the parameters used throughout the paper. Section 3 defines $N2R$ with one as well as more chord lengths, and results related to $N2R$ are presented. Similarly, Section 4 defines Chordal Rings and related results. Section 4 also compares the impact of using more chord lengths in $N2R$ and Chordal Rings. Section 5 concludes the paper and points towards potential future research directions.

2. Preliminaries and Methods

A network structure or topology S is a set of nodes and a set of lines, where each line interconnects two nodes. Lines are bidirectional, so if a pair of nodes (u, v) is connected, so is (v, u). A structure can be considered a model of a network, abstracting from specific physical conditions such as node equipment, medias and wiring. The definition of a structure is similar to that of a simple graph in graph theory. A path between two distinct nodes u and v is a sequence of nodes and lines: $(u = u_0), e_1, u_1, e_2, u_2, \ldots, u_{n-1}, e_n, (u_n = v)$, so that every line e_i connects the nodes u_{i-1} and u_i. The length of a path corresponds to the number of lines it contains, so in the case above the path is of length n. The distance between a pair of nodes (u, v) corresponds to the length of the shortest path between them and is written $d(u, v)$. This paper considers only connected structures, i.e. between every pair of nodes there exists a path. The size of a structure/network corresponds to the number of nodes it contains.

The topologies of the paper are compared by calculating average distance and diameter for all topologies with up to 2000 nodes. The parameters reflect important factors such as delay and load on switches/routers. The definitions are given as:

- Average distance: The average of $d(u, v)$ taken over all pairs of distinct nodes.
- Diameter: The maximum of $d(u, v)$ taken over all pairs of distinct nodes.

All results obtained in the paper are obtained by calculating average distances and diameters of the different graphs, using standard shortest-path algorithms. A main challenge has been to select (for each class of topology) only one topology given the number of nodes. This is especially difficult when three chord lengths are introduced, because there exist many different graphs with different combinations of chord lengths. Since the graphs are of limited size (2000 nodes), and due to the high degree of symmetry, it has been possible to calculate both average distance and diameter for all possible graphs. So, given the number of nodes it was for each type of topology possible to choose the one with lowest diameter and average distance (in case it was not possible to minimize both parameters the one with lowest diameter was chosen).

3. N2R

First the definition of the traditional $N2R$ structure, with one chord length, is given. Let p and q be positive integers, so that p is even, $p \geq 3$, $q < \frac{p}{2}$ and $gcd(p, q) = 1$. These values of p and q define a $N2R(p; q)$ structure S as follows: S consists of two rings, an outer ring and an inner ring, each containing p nodes. The nodes of the outer ring are named $o_0, o_1, \ldots, o_{n-1}$ and the nodes of the inner ring are named $i_0, i_1, \ldots, i_{n-1}$. Thus, S contains $2p$ nodes. For each i such that $0 \leq i \leq n - 1$ there exists a line between each of the following pairs of nodes:

- $(o_i, o_{i+1(mod\ p)})$ (lines of the outer ring)
- $(i_i, i_{i+q(mod\ p)})$ (lines of the inner ring)
- (o_i, i_i) (lines connecting the two rings)

The classical Double Ring with $2p$ nodes obviously corresponds to $N2R(p; 1)$. It is similar to the definition of the Generalized Petersen Graph[Frucht et al. 1971], except for the restriction of $gcd(p, q) = 1$. For an example of a traditional $N2R$ structure see Figure 1.

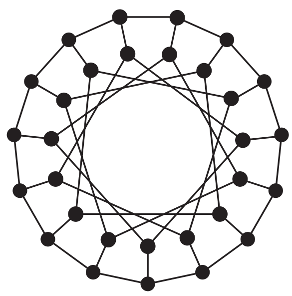

Figure 1. N2R(15;4).

For the generalization with two chord lengths, the parameters q_1 and q_2 are used instead of q. It is assumed that $gcd(p, q_1) = gcd(p, q_2) = 2$. This will ensure that there are two inner rings, possibly with different jump lengths, which is an essential property of the $N2R$ topologies ($N2R$ means Network of 2 Rings). We now write $N2R(p; q_1/q_2)$.

The definition is similar to the traditional $N2R$ as given above, except for the definition of lines in the inner ring where a line exists between each of the following pairs of nodes:

- For i even, $(i_i, i_{i+q_1(mod\ p)})$
- For i odd, $(i_i, i_{i+q_2(mod\ p)})$

The definition is based on the fact that both q_1 and q_2 must be even numbers. Note also that this implies that the two inner rings each contain $\frac{p}{2}$ nodes. Introducing more chord lengths is rather straightforward; for x chord lengths, we simply assume that $gcd(p, q_1) = gcd(p, q_2) = \ldots = gcd(p, q_x) = x$, and change the lines of the inner ring so that there exists a line between each of the following pairs of nodes:

- For $i \equiv 0(mod\ x)$, $(i_i, i_{i+q_1(mod\ p)})$
- For $i \equiv 1(mod\ x)$, $(i_i, i_{i+q_2(mod\ p)})$
- $\cdots$
- For $i \equiv x-1(mod\ x)$, $(i_i, i_{i+q_x(mod\ p)})$

With x chord lengths, we denote the topology $N2R(p; q_1; q_2; \ldots; q_x)$. Figure 2 provides an example of how the definition works. In the example there are really three classes of chords, but two of them are of equal length.

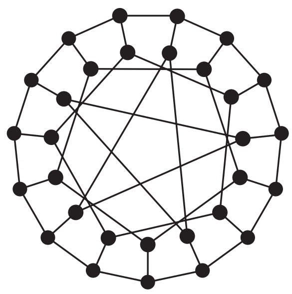

Figure 2. N2R(15;6;3;3).

3.1. Results

Figures 3-4 show the average distances and diameters of $N2R$ with one, two and three different chord lengths. While for large topologies three different chord lengths reduce the distances, the differences between using two or three chord lengths are not huge. For example, for topologies with 1992 nodes, the average distances are 16.39, 9.87 and 9.15 for one, two and three chord lengths respectively. The pattern is the same for diameters; these are 26, 14 and 13 respectively. The distances vary only insignificantly depending on from which node it is measured.

4. Degree Three Chordal Rings

Again, first the definition of Chordal Rings with only one chord length is given. Let w be an even integer such that $w \geq 6$, and let s be an odd integer, such that $3 \leq s \leq \frac{w}{2}$. w and s then define $CR(w; s)$ with w nodes labeled $u_0, \ldots, u_{w-1}$. For $0 \leq i \leq w-1$ there exists a line between each of the following pairs of nodes:

- $(u_i, u_{i+1(mod\ w)})$
- $(u_i, u_{i+s(mod\ w)})$, for i even.

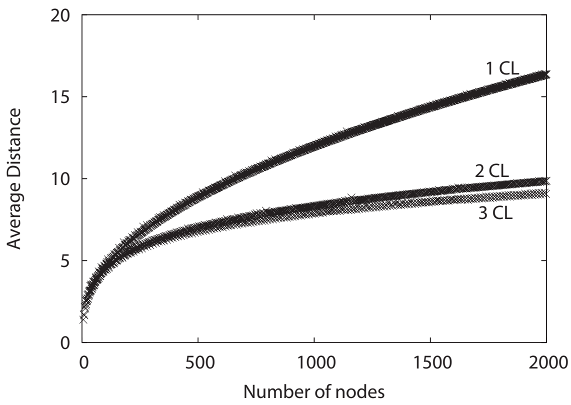

Figure 3. Average distances in N2R with 1, 2 and 3 different chord lengths.

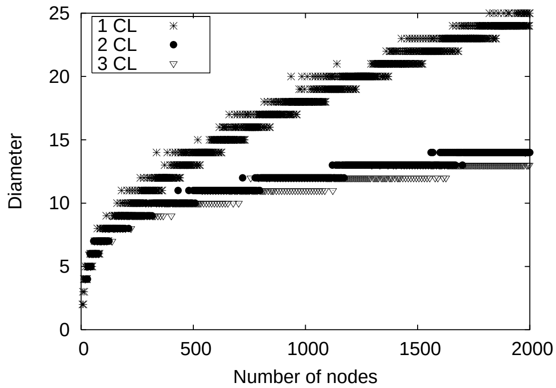

Figure 4. Diameters in N2R with 1, 2 and 3 different chord lengths.

An example of a Chordal Ring with one chord length is shown in Figure 5.

More chord lengths can be introduced in different ways, and is different and less straight-forward than for the $N2R$ topologies because each node is connected to only one chord. We propose to introduce different chord lengths in a way that preserve high levels of regularity and symmetry.

As a first step, neighbour pairs of nodes are divided into a number of groups corresponding to the number of different chord lengths. For example, in case of 3 chord lengths assume that the nodes are numbered $1, 2, \ldots, w$ as given above. We require w to be divisible by two times the number of chord lengths, i.e. in this case divisible by 6. Then we let node 1 and 2 belong to group 1, node 3 and 4 belong to group 2, node 5 and 6 belong to group 3, node 7 and 8 belong to group 1 etc. An example of such a division is given in Figure 6.

Each pair now consists of one even numbered and one odd numbered node. The chords are then introduced, with each group of pairs being assigned a chord length. For group j a chord of uneven length s_j is assigned, so that for i even, node u_i belonging to group j is connected to the node $u_{i+s_j (mod\ w)}$, where it is a condition that this node also belongs to group j (note that this node is odd numbered). This puts some constraints on the chord lengths, but it can be verified that the set of chord lengths expressed by $k{\cdot}2{\cdot}(number\ of\ groups){+}1$, where k is a positive integer, forms the set of possible chord

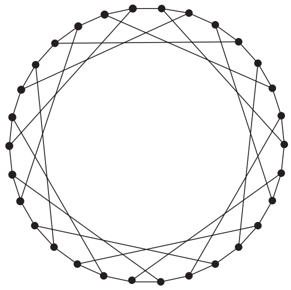

Figure 5. CR(30;7).

lengths that may not exceed one fourth of the number of nodes in each group (i.e. it may not exceed $\frac{w}{4\cdot(number\ of\ chord\ lengths)}$. (more values of k could be allowed, but the current definition allows all possible connections of pairs). A Chordal Ring with x different chord lengths $s_1, \ldots, s_x$ is written $CR(w; s_1; \ldots; s_x)$.

In Figure 6 the chords, of length 7 corresponding to $k = 1$, are shown for one group of pairs. In Figure 7 the chords for the two other groups of pairs are shown. These are chosen to be 7 (again $k = 1$) and 13 ($k = 2$) respectively.

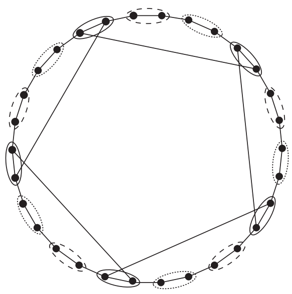

Figure 6. Dividing the nodes of the Chordal Ring into 3 groups, which will have different chord lengths. The chords of one group of nodes are shown.

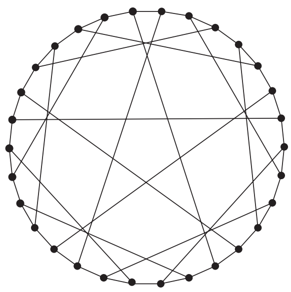

Figure 7. A Chordal Ring with 3 classes of chords, CR(30;7;7;13).

4.1. Results

Figures 8-9 show the average distances and diameters of Chordal Rings with one, two and three different chord lengths. It shows that the distances are significantly reduced by introducing more chord lengths, and the distances are shorter with three chord lengths than with two. This is especially so for large topologies. As for $N2R$ the difference between having one or two chord lengths is bigger than the difference between having two or three. However, the impact of having three chord lengths seems more significant than for $N2R$. For an example, consider again the topologies with 1992 nodes. The average distances here are 24.31, 11.68 and 9.29 for one, two and three chord lengths respectively, and the diameters 38, 18 and 13. As for $N2R$ it turns out that the distances vary only insignificantly depending on from which node it is measured.

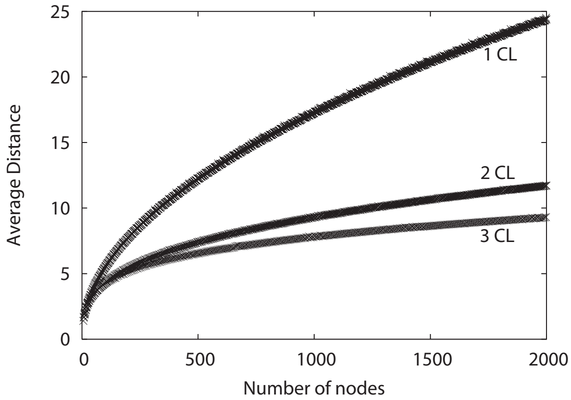

Figure 8. Average distances in Chordal Rings with 1, 2 and 3 different chord lengths.

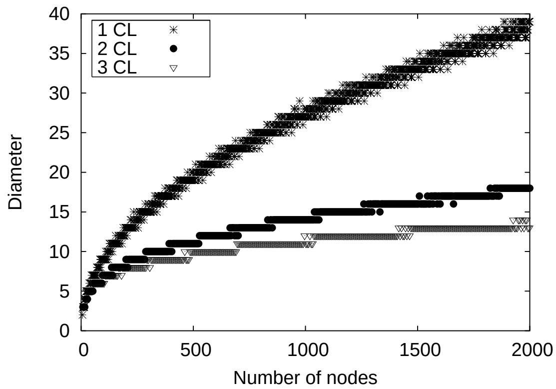

Figure 9. Diameters in Chordal Rings with 1, 2 and 3 different chord lengths.

With the calculations in place it is interesting to compare the impact of having three different chord lengths in $N2R$ and Chordal Rings. Figures 10-11 compare the average distances with one and three chord lengths respectively. It is seen that with only one chord length $N2R$ has significantly shorter distances than Chordal Rings, particularly for large topologies. With three chord lengths they are nearly indistinguishable. Using topologies with 1998 nodes as an example, Chordal Rings and $N2R$ with one chord length have average distances 24.51 and 16.35 respectively, while the numbers are 9.31 and 9.07 for three different chord lengths.

Similar results are obtained for the diameters, and shown in Figures 12-13. Taking 1998 nodes as an example again, the diameters of Chordal Rings and $N2R$ are for one chord length 39 and 24 respectively, and for three chord lengths 13 and 13 respectively.

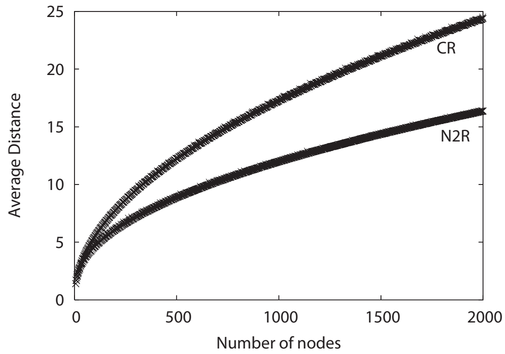

Figure 10. Average distances in Chordal Rings and N2R with one chord length.

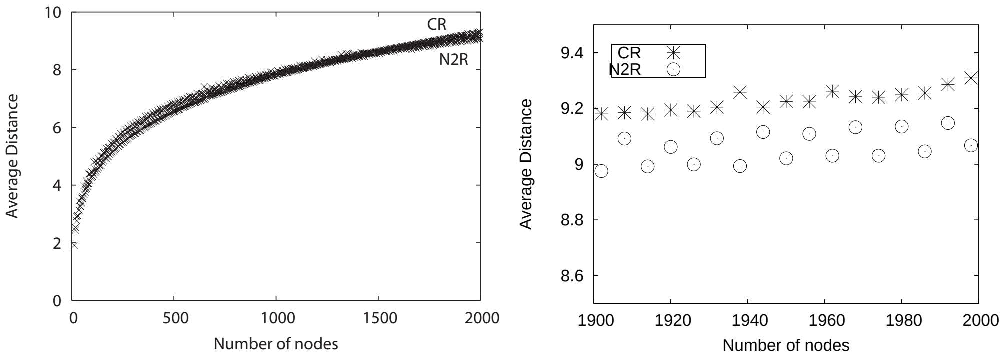

Figure 11. Average distances in Chordal Rings and N2R with three chord lengths. The second plot shows the results for 1900-2000 nodes for increased readability.

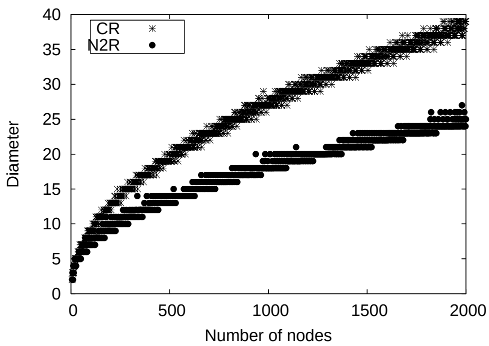

Figure 12. Diameters in Chordal Rings and N2R with only one chord length.

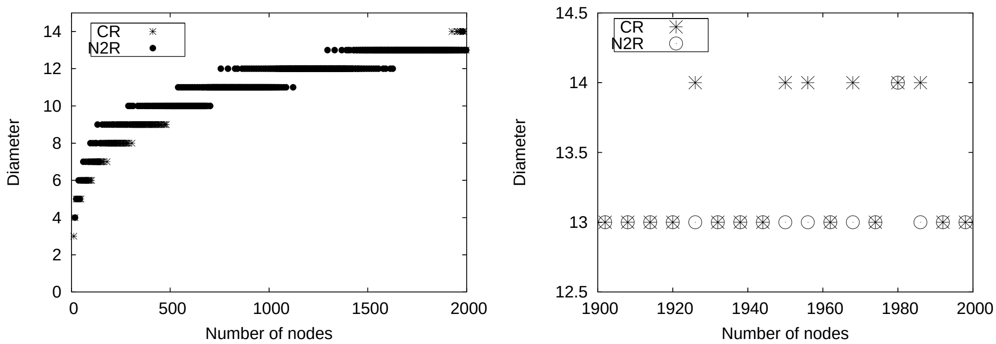

Figure 13. Diameters in Chordal Rings and N2R with three chord lengths.

5. Conclusion and Further Work

Simple 3-regular topologies er interesting for the design of reliable networks, both for optical and physical layers. In line with this, previous research has analysed $N2R$ networks and Degree Three Chordal Rings and found them to have a number of good properties, including low distances. A recent contribution by the authors showed that using two chord lengths instead of one in $N2R$ could reduce the distances significantly, especially for large networks.

In this paper we take this generalization one step further, and show how distances in $N2R$ can be reduced even more by using three different chord lengths. All possible topologies with up to 2000 nodes were evaluated in terms of average distance and diameter. For topologies with 1992 nodes, the average distances are 16.39, 9.87 and 9.15 for one, two and three chord lengths respectively. The diameters are 26, 14 and 13 respectively.

Furthermore we propose a construction of Degree Three Chordal Rings with different chord lengths, and evaluate the impact on distances of using two or three different chord lengths. This is also done for all possible topologies with up to 2000 nodes. This construction leads to significant reduction of average distance and diameters in these graphs. Actually the impact of using three different chord lengths is more significant for the Chordal Rings than for the $N2R$: For topologies with 192 nodes the average distances are here 24.31, 11.68 and 9.29 for one, two and three chord lengths respectively, and the diameters 38, 18 and 13.

The results are interesting because they show how it is possible to build graphs with shorter average distance and diameter, making it potentially possible to design networks with shorter delays and less load on switching/routing equipment. They could be used at the optical levels in fiber networks, or at the physical level of any wired network technology if appropriate ways of deployment can be found. The gain is almost for “free” since the topologies have the same properties with respect to numbers of nodes, connectivity and degree. The main drawback is that the networks are slightly more complicated and less symmetric. Concerning the symmetry we found out that the distance characteristics only vary insignificantly when measured from the different nodes in both Chordal Rings and $N2R$.

In order to take advantage of these potentials it is necessary to work further on how to apply the topologies in real-world networks on optical or physical levels. The fact that both topologies are based on rings should be taken advantage of.

The proposed constructions of both Chordal Rings and $N2R$ are also useful for even more different chord lengths, which could reduce the distances further for very large topologies. This raises the challenge of how to select these chord lengths. For up to three chord lengths, and 2000 nodes, the results of this paper could be obtained by simply calculating the distances in all possible graphs. For more complex calculations a more analytical approach for selecting the chord lengths will be needed.

Another direction of further research could be to revise the definitions of Chordal Rings and $N2R$ with different chord lengths. For $N2R$ we believe that lower distances could be obtained by not requiring the inner rings to be connected rings, and for Chordal Rings it could be interesting to define the different chord lengths in other ways.

References

Bujnowski, S., Dubalski, B., and Zabludowski, A. (2004). The evaluation of transmission ability of 3rd degree chordal rings with the use of adjacent matrix. *Proceedings of the 7th INFORMS Telecommunications Conference, Miami, USA*.

Dekker, A. H. and Colbert, B. D. (2004). Network robustness and graph topology. *Proceedings of ACS04, the 27th Australasian Computer Science Conference, Dunedin, New Zealand*, pages 359–368.

Frucht, R., Graver, J. E., and Watkins, M. E. (1971). The groups of the generalized petersen graphs. *Proceedings of the Cambridge Philosophical Society*, 70(2):211–218.

Kotsis, G. (1992). *Interconnection Topologies and Routing for Parallel Processing Systems*. Technical Report, ACPC/TR 92-19. ACPC, Austrian Center for Parallel Computation.

Lynch, C. (2000). Why broadband really matters: Applications and architectural challenges. *Educause Quarterly*, 23(2):59–62.

Madsen, O. B., Knudsen, T. P., and Pedersen, J. M. (2003). SQoS as the base for next generation global infrastructure. *Proceedings of IT&T 2003, Information Technology and Telecommunications Annual Conference 2003, Letterkenny, Ireland*, pages 127–136.

Madsen, O. B., Nielsen, J. D., and Schiøler, H. (2002). Convergence. *Proceedings of RTLIA 2002, 1st International Workshop on Real-Time LANs in the Internet Age, Vienna, Austria*, pages 101–104.

Pedersen, J. M. (2005). *Structural Quality of Service in Large-Scale Networks*. PhD Thesis, Aalborg University.

Pedersen, J. M., Knudsen, T. P., and Madsen, O. B. (2004). Comparing and selecting generalized double ring network structures. *Proceedings of IASTED CCN 2004, The Second IASTED International Conference on Communication and Computer Networks, Cambridge, USA*, pages 375–380.

Pedersen, J. M., Knudsen, T. P., and Madsen, O. B. (2005a). Reliability demands in ftth access networks. *Proceedings of IEEE/ICACT 2005, The 7th International Conference on Advanced Communication Technology, Korea*, 2:1202–1207.

Pedersen, J. M., Riaz, T. M., and Madsen, O. B. (2005b). Distances in generalized double rings and degree three chordal rings. *Proceedings of IASTED PDCN 2005, The IASTED International Conference on Parallel and Distributed Computing and Networks, Innsbruck, Austria*, pages 153–158.

Pedersen, J. M., Riaz, T. M., Nielsen, R. H., and Madsen, O. B. (2006). A generalization of the generalized petersen graphs and n2r network structures. *Proceedings of IASTED CCN 2006, The IASTED International Conference on Communication and Computer Networks, Lima, Peru*.

Stojmenovic, I. (1997). Honeycomb networks: Topological properties and communication algorithms. *IEEE Transactions on Parallel and Distributed Systems*, 8(10):1036–1042.