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Reliability bounds for networks with statistical dependence

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ABSTRACT

Many bounds for the all-terminal reliability of a network have been proposed, but most assume that link failures are statistically independent. This paper develops a lower bound for the all-terminal reliability of a network when statistical dependence of link failures occurs.

1. Introduction

A computer network is typically modelled as a *probabilistic graph;* the undirected edges of the graph represent bidirectional communication links, and the nodes represent sites of the network. Failure probabilities are associated with each edge of the network. In this setting, the *all-terminal reliability* of the network is the probability that the operational edges provide communication paths between all pairs of nodes. Most studies have assumed that link failures are statistically independent; the resulting measure is a unique probability that the network is operational. Even with the assumption of statistical independence, all-terminal reliability is hard to compute [6], which has led to investigations of efficiently computable bounds [2,3,5].

When statistical independence does not hold, little is known. Hailperin [4] developed a linear programming model, which Zemel [7] used to obtain the best possible bounds when the only information given concerns failures of individual links and the dependencies between link failures are unknown. These are called the *first-order* bounds. Improvements on these bounds can only be obtained by exploiting information about the failure probabilities of pairs of links; the result of employing this information in Hailperin's model gives the *second-order* bounds. Assous [1] developed second-order bounds for the two-terminal reliability problem using Hailperin's model; to obtain the lower bound, however, one must solve a quadratic programming problem, and no computationally efficient technique is known for this. Hence, Assous produces a simple heuristic technique for producing a lower bound using graph-theoretic techniques and the second-order information. In section 3, we develop a similar heuristic lower bound in the case of all-terminal reliability.

2. First-Order Bounds

In the weakest model, we assume that each edge e_i has a success probability p_i satisfying $a_i \le p_i \le b_i$. No other information about failures is known, and no assumption of statistical independence is made. The network is *coherent,* however, in that the failure of an edge cannot make a failed network operational. In this context, Hailperin [4] showed that the tightest lower bound on all-terminal reliability is obtained by solving the linear program

$$\textit{minimize} \qquad \sum_{S \epsilon F} Y_S$$

subject to

$$\sum_{S | i \epsilon S} Y_S \le b_i, \quad 1 \le i \le e$$

$$\sum_{S | i \epsilon S} Y_S \ge a_i, \quad 1 \le i \le e$$

$$\sum_{S \subseteq \{1,\dots,e\}} Y_S \le 1$$

and nonnegativity constraints. In this linear program, F is the set of operational configurations of the network, and S varies over all configurations, both failed and operational.

The direct application of Hailperin's model is computationally intractable, since there are an exponential number of variables and constraints. Zemel [7] showed how to solve an equivalent problem efficiently, and later Assous [1] developed a simple computational method which we describe next. Assous showed that the minimum value L achieved by Hailperin's linear program satisfies

$$L = max(0, 1 - \min_{S \epsilon F^*} \sum_{j \epsilon S} (1 - a_j))$$

where F^* is the set of all spanning trees of the network. Thus L is simply the weight of a minimum weight spanning tree of the network obtained using $1 - a_j$ as edge weights.

Although computationally appealing, the first-order bounds are very poor indeed for any practical purposes. For example, when $p = 0.9$ for each link and the network has more than ten nodes, the lower bound is zero. One cannot fault the bounds for this, as there is no hope of obtaining a better bound unless additional information is provided. Nevertheless, the need for better bounds is clear.

3. Second Order Bounds

In an effort to improve the first order bounds, we assume that in addition to the previous model, for every pair i,j of edges, we have bounds on q_{ij}, the probability that edges i and j fail simultaneously. In particular, we suppose that $\underline{b}_{ij} \leq q_{ij} \leq \bar{a}_{ij}$. We continue to use first-order constraints, $\underline{b}_i \leq q_i \leq \bar{a}_i$, where $q_i = 1-p_i$.

Once again, Hailperin's model can be used to set up a linear program; here, however, there is no easy way to circumvent the exponential size of the linear program. We therefore resort to heuristic techniques. Given the network N we first find the most reliable spanning tree T of N. For each edge $e=(x,y)$ of T, let L_e be an upper bound on the probability that x and y have no operational path between them. Then a lower bound on the reliability is given by

$$1 - \sum_{e \in T} L_e \qquad (*)$$

To obtain the first-order bound, we simply observe that $L_e \leq \bar{a}_e$. However, second order information can be used to improve this.

For each $e \in T$, $e=(x,y)$, construct a network N_e by deleting each edge of T from N and setting the weight of each edge f to $\bar{a}_{ef}$. Find a minimum weight path P from x to y, if one exists. Let W_e be the weight of the path P if one is found, ∞ otherwise. Now observe that $L_e \leq W_e$. In fact, W_e is an upper bound on the probability that edge e and path P fail simultaneously. One might consider including further $x-y$ paths, but then third- and higher-order information would be required to bound L_e; since only one path can be chosen, we select the most reliable. Combining the two constraints, we have $L_e \leq min(\bar{a}_e, W_e)$. Substituting the values of L_e obtained into (*) above, we obtain a lower bound on the all-terminal reliability. This bound can, of course, be no worse than the first-order bound, and is typically much better.

We have tested the new lower bound on a number of networks; to do so, we assume that pairs of failures are statistically independent, but make no assumptions about triples, quadruples, and so on.

The results of these tests are not surprising. Once statistical independence cannot be assumed, the bounds are significantly weaker than bounds which assume statistical independence. For example, we computed bounds for a 6-regular, 6-connected graph which is a "chordal ring"; it has nodes numbered 0 through 11, and edges from node x to nodes $x+1, x+3, x+5$ for every x (arithmetic modulo 12). For all $p < .9$, the first order bound delivers a value of 0; however, the second order bound improves on this dramatically. Nevertheless, the second order bound is 0 for all $p < .82$. Comparing these bounds with bounds which do assume statistical independence, they provide much weaker bounds on the reliability. To illustrate this, we compare with the bounds from [3]:

Twelve Node Chordal Ring			
p	First Order	Second Order	[3]
.8	0.0	0.0	.995995
.85	0.0	.247595	.999265
.9	0.0	.660996	.999934
.92	.12	.788797	.999983
.94	.34	.881198	.999997
.96	.56	.947198	$1.0 - \epsilon$

4. Concluding Remarks

The most striking conclusion is that when one abandons the assumption of statistical independence of edge failures, the usefulness of bounds to estimate reliability is minimal. Nevertheless, in certain contexts no information about high-order correlations is available, and the assumptions about statistical correlations are dangerous. In these contexts, the second order bound developed here proves to be a definite asset; the improvement in the accuracy of the bound makes the collection of second order information worthwhile.

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